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Projection and elevation of ordinal diagrams

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§0. Introduction

The system of ordinal diagrams was invented by G. Takeuti as a means for the consistency proofs; that is, the consistency of a subsystem of analysis is reduced to the accessibility of ordinal diagrams. It is therefore of primary importance that we establish an accessibility proof of ordinal diagrams in its strict sense. Such attempts have been made over the years, but they were not entirely satisfactory from our standpoint.

Let $\mathcal{J}=(J,<)$ be a concretely given linearly ordered structure. An accessibility proof of $\mathcal{J}$ consists in presenting a "concrete" method to demonstrate that there be no infinite decreasing sequence

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from J with respect to <, and J is said to be <-accessibile if there is such an accessibility proof.

I have recently succeeded in an accessibility proof of ordinal diagrams, which is concrete and yet admits a strict formulation (if one so wishes). In the present notes we place emphasis on "projections" and "elevations" of ordinal diagrams, which are the essential ideas in this accessibility proof.

§1. Basics of ordinal diagrams

The notions concerning the ordinal diagrams are to be explained first for those who are not acquainted with them.

Definition 1. Let I, A and S be non-empty well-ordered sets.

1) We define the set of (ordinal) diagrams $O=O(I, A, S)$.

(1) Each element of S is a connected diagram of O.

(2) If $i \in I$, $a \in A$ and $\alpha \in O$, then (i, a, α) is a connected diagram of O.

(3) If $m \geq 2$ and $\alpha_1, \cdots, \alpha_m$ are connected diagrams of O, then $\alpha_1 \# \cdots \# \alpha_m$ is a non-connected diagram of O. Each α_ℓ is call a component of this diagram.

2) For α, $\beta \in O$, $\alpha=\beta$ if α and β are defined in the same manner except the orders of the components in the applications of (3).

Note. In the subsequent discussion, an occurrence of an object x in a diagram will simply be referred as x unless confusion is feared.

3) If β is a part of α, then β is said to be i-active in α if β can be reached from outside within α without encountering any element of I below i.

4) β is called an i-section of α if there is a subdiagram of α of the form (i, a, β) which is (i+1)-active in α.

5) If $j \in I$, then $j_0(j, \alpha, \beta)$ will denote the least element i of $I^{\cup}\{\infty\}$ such that $i=\infty$ or $i>j$ and there is an i-section of α and/or β.

6) We define $<_i$, the i-order of O for i $I^{\cup}\{\infty\}$.

6.1) The order of S is retained.

6.2) # is regarded as the natural sum.

6.3) The elements of S are below any diagram with an application of (2) with respect ot any $<_i$.

6.4) If $\alpha \equiv (j, a, \gamma)$ and $\beta \equiv (k, b, \delta)$, then $\alpha <_\infty \beta$ if one of i)~iii) below holds.

i) $j<k$ (in I).

ii) $j=k$ and $a<b$ (in A).

iii) $j=k$, $a=b$ and $\gamma <_j \delta$.

6.5) If $i \in I$, and α and β are connected and compound, then $\alpha <_i \beta$ if one of i) and ii) below holds.

i) There is an i-section σ of β such that $\alpha \leq_i \sigma$.

ii) $\alpha <_j \beta$, and $\delta <_i \beta$ for every δ an i-section of α, where $j=j_0(i, \alpha, \beta)$.

Remark. Let us comment on the basic sets I, A nad S. The sets I and A are in fact assumed to be accessible sets, while S is regarded as a parameter for the accessible sets. That is, if an accessible set J is concretely presented, then we can substitute it for S to obtain a concrete system O(I, A, J), but the elementary theory of O can be develeped with S a parameter.

Proposition 1. 1) = is an equivalence relation.

2) <i is a linear order for every i.

3) If β is an i-section of α, then $\beta <_i \alpha$.

4) α is a "successor" in $(O, <_i)$ (for any i) if α has o, the least element of S, as a component; α is "limitary" otherwise.

5) O(I, A, S) is "well-ordered" by $<_i$ for every $i \in I^{\cup}\{\infty\}$.

This is a set-theoretical fact, and should be distinguished from the notion of "accessibility".

In the second reference, we established the following

Theorem ([2]). There is a concrete method, say $\mathcal{F}$, relative to S, to construct a fundamental sequence for every pair (α, i), α a limitary diagram and i an element of I. That is, $\mathcal{F}$ produces a sequeme from O which converges to α from below with respect to i, presuming that such a method exists for S.

$\mathcal{F}$ is concrete in the sense that it is "primitive recursive" relative to the fundamental method of S.

§2. Speculation

Before we go into the technical defails, let us speculate a little on the notion of accessibility.

Takeuti originally proved the accessibility of O by making use of a subset of it, which he named F_i, for each i an element of $I \cup \{\infty\}$. He later became declined to accept F_i as a concrete object, and he and I proposed another version of the accessibility proof in [3], in which the theory of fundamental sequences just stated above and the notion of strong accessibility stand essential. Since the construction of fundamental sequences is finitary (see [2]), the problem of this approach can be pinned down to the notion of strong accessibility, which is defined in terms of arbitrary well-ordered sets.

Now, I believe that the problem is not whether or not it is acceptable to consider F_i; we can first introduce it as an abstract concept, work out an accessibility proof by using it as an auxiliary means, and later give substance to it. This is justified since F_i is a well-defined concept.

As will be seen subsequently, I have returned to F_i according to the belief above. The details are seen in [5]. In fact this attitude of mine has been motivated by Takeuti's recent work [4], in which he also returned to F_i.

§3. Definitions and remarks

We now present the exact definitions of the objects we are to need, and then give a brief account of our work.

Definition 2. Suppose $i \in I^{\cup}\{\infty\}$.

$F_i(S) = \{\alpha \in O(S);\ \forall j<i$ (every j-section of α is j-accessible in F_j)$\}$

$G_i(S) = G_i = \{\alpha \in F_i(S);\ \alpha$ is i-accessible in $F_i(S)\}$

$H_i(S) = \{(k,\ b,\ \beta);\ k<i$ and $\beta \in G_k\}$

$\gamma[i]$: a new symbol corresponding to γ if $\gamma \in H_i$.

$J_i(S) = \{\gamma[i];\ \gamma \in H_i\}$

$\prec_i$: the order of J_i induced from $<_i$.

$I\langle i\rangle = \{j \in I;\ j \geq i\}$

$O(i) = O(I\langle i\rangle,\ A,\ S * J_i)$

$D_i = \{\kappa \in O(i);\ \kappa$ is i-accessible in $O(i)\}$

For an $\alpha \in F_i$, $\alpha[i]$, the i-projection of α, is defined to be the figure obtained from α by replacing in it each i-active element of H_i by its corresponding symbol in J_i. For a κ in $O(i)$, the unique $\alpha \in F_i$ such that $\alpha[i]=\kappa$ is called the i-elevation of κ, and is denoted by $\kappa\{i\}$. (Such an α exists.)

If $\alpha \in F_i$, then we say that α is i-fit (originally α was said to be an i-fan); if $\alpha \in G_i$, then we say that α is i-grounded.

Outline. We can show that the i-groundedness property is reduced to the $<_i$-accessibility in $O(i)$, that is $\alpha \in G_i$ if and only if $\alpha[i] \in D_i$. Through certain constructions induced by the fundamental sequences in $O(j)$ for some $j<i$, we can demonstrate that every diagram is i-grounded, and is hence $<_i$-accessible (for every i). Notice that j is the least element of $I^{\cup}\{\infty\}$ in $O(j)$ even if it is not in the original system O; this simple fact has a decisive effect on the entire argument.

§4. Elementary properties of projection and elevation

Proposition 2. 0) G_i is i-accessible, and J_i is $\prec_i$-accessible.

1) If $\alpha \in F_i$, then $\alpha[i] \in O(i)$ and $\alpha[i]\{i\} = \alpha$.

2) The i-projection yields an isomorphism between $(F_i, <j)$ and $(O(i), <j)$ for every $j \geq i$.

3) If $\alpha \in F_i$, then $\alpha \in G_i$ if and only if $\alpha[i] \in D_i$.

4) For any α, $\beta \in F_{i+1}$, $\beta <_i \alpha$ and $\alpha <_{i+1} \beta$ imply $\beta \in G_i$.

5) The fundamental sequences can be constructed in $O(i)$.

Of these, 4) will be used in the negative form:

$$\alpha,\ \beta \in F_{i+1},\ \beta <_i \alpha,\ \beta \notin G_i \rightarrow \beta <_{i+1} \alpha.$$

There are other negative proof procedures such as: to introduce a contradiction from the negative assumption $\kappa \notin D_i$. This is for the sake of descriptive economy, and the entire proof is so designed that the negative arguments can be eliminated. This can be done by the functional interpretation of the proof procedure, but it is a problem left for the future.

§5. The key propositions

Definition 3. 1) Suppose $p<i$ and $\kappa \in O(p+1)$. κ is said to be (p, i)-free if $\forall j(p<j<i \vdash j$ does not occur in $\kappa)$.

2) Suppose $p<i$ and $\kappa \in O(p+1)$. When κ is (p, i)-free, κ is said to be (p, i)-secure if κ is j-accessible in $O(j)$ for every j such that $p<j<i$.

Theorem. 1) (The first key proposition) If $\kappa \in O(i)$, κ is connected, $\kappa\{i\} \in F_{i+1}$, $\kappa\{i\}[i] \in D_{i+1}$, then $\kappa \in D_i$.

2) (The second key proposition) (i)~(v) below imply that κ is (p, i)-escure.

(i) i is limitary.

(ii) $\forall q<i \forall f \in Dq \forall j<q (f\{q\}[i] \in D_j)$

(iii) $p<i$

(iv) $\kappa \in O(p+1)$ and κ is (p, i)-free.

(v) $\kappa \in D_i$.

From 1) and 2) follows that an i-grounded diagram is j-grounded for every $j<i$, and this fact is essential for the accessibility proof.

§6. Final remarks

We did not explicitly state that "there be concrete methods" in defining various notions; such methods underlie the entire argument.

Our proof is an improvement over [3] also in the sense that in a way the "arbitrary well-ordered sets" in the definition of strong accessibility have been replaced by a concrete accessible set J_j for each j.

Let us emphasize that we have not given an alternative (accessibility) proof to the preceding ones, but our approach serves as a conclusive version of the accessibility proof of ordinal diagrams.

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